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LINEARITY OF ARTIN GROUPS OF FINITE TYPE*

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ABSTRACT

Recent results on the linearity of braid groups are extended in two ways. We generalize the Lawrence Krammer representation as well as Krammer's faithfulness proof for this linear representation to Artin groups of finite type.

1. Introduction

Recently, both Bigelow [Bigelow] and Krammer [Krammer] proved that the braid groups are linear. The braid group on $n+1$ braids is the Artin group of type A_n. This paper extends the result to all Artin groups whose types are finite, that is, belong to finite Coxeter groups.

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THEOREM 1.1: *Every Artin group of finite type is linear.*

Linearity of a group means that it has a faithful linear representation. A standard argument reduces the proof to Artin groups whose types are finite and irreducible. We focus on the Artin groups of type A, D, E. Since the other Artin groups of finite and irreducible type can be embedded in these (cf. [4]), it will suffice for a proof of Theorem 1.1 to exhibit a faithful representation for each of the groups A, D, E. The theorem below provides more information about the representation found.

Throughout the paper, we fix a Coxeter matrix M of dimension n, and denote by B the Artin group of type M. This means that B is the group generated by n elements $s_1, \ldots, s_n$ subject to the relations

$$\underbrace{s_i s_j s_i \cdots}_{\text{length } M_{ij}} = \underbrace{s_j s_i s_j \cdots}_{\text{length } M_{ij}} \tag{1}$$

for $1 \leq i < j \leq n$. The Coxeter system of type M is denoted by (W, R) with R consisting of the images r_i of s_i $(i = 1, \ldots, n)$ under the natural homomorphism from B to the Coxeter group W. We use the standard facts and some terminology of root systems as treated for example in [2]. We shall be working solely with Artin groups of finite type, so W is assumed finite, and W has a finite root system Φ in $\mathbb{R}^n$. We shall denote by $\alpha_1, \ldots, \alpha_n$ the fundamental roots, corresponding to the reflections $r_1, \ldots, r_n$, respectively, and by Φ^+ the set of positive roots:

$$\Phi^+ = \Phi \cap \bigoplus_{1 \leq i \leq n} \mathbb{R}_{\geq 0} \alpha_i.$$

Then Φ is the disjoint union of Φ^+ and $\Phi^- = -\Phi^+$. If $M = A_n$ $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, or E_8, we say that B is of type A, D, E.

The coefficients of our representation will be taken in the ring $\mathbb{Z}[r, t, r^{-1}, t^{-1}]$, and we write V for the free module over that ring with generators x_β indexed by $\beta \in \Phi^+$.

THEOREM 1.2: *Let B be an Artin group of type A, D, E. Then, for each $k \in \{1, \ldots, n\}$ and each $\beta \in \Phi^+$, there are polynomials $T_{k,\beta}$ in $\mathbb{Z}[r]$ such that the following map on the generators of B determines a representation of B on V:*

$$s_k \mapsto \sigma_k = \tau_k + tT_k,$$

where τ_k is determined by

$$\tau_k(x_\beta) = \begin{cases} 0 & \text{if } (\alpha_k, \beta) = 2 \\ r x_{\beta-\alpha_k} & \text{if } (\alpha_k, \beta) = 1 \\ x_\beta & \text{if } (\alpha_k, \beta) = 0 \\ (1-r^2)x_\beta + r x_{\beta+\alpha_k} & \text{if } (\alpha_k, \beta) = -1 \end{cases}$$

and T_k is the linear map on V determined by $T_k x_\beta = T_{k,\beta} x_{\alpha_k}$ on the generators of V. If r is specialized to a real number r_0, $0 < r_0 < 1$, in $V \otimes \mathbb{R}$, we obtain a faithful representation of B on the resulting free $\mathbb{R}[t, t^{-1}]$-module V_1 with basis x_β ($\beta \in \Phi^+$).

The proof of this theorem is based on Krammer's methods. More specifically, we generalize the Lawrence Krammer representation [6, 7] as well as Krammer's faithfulness results from braid groups to Artin groups corresponding to a spherical root system with a single root length. The difficulties in the proof come in the proper definition of the $T_{k,\beta}$. These are determined by Algorithm 3.4. The algorithm has been implemented in the computer algebra package Maple and has been used to construct the representations of B for all M of type A, D, E, and rank at most 10. In Example 3.8 we verify that, for type A, the representation of Theorem 1.2 is indeed the Lawrence Krammer representation.

In Section 2 we recall basic and useful properties of Artin groups, including generalizations of some of Krammer's results on braid groups. Section 3 introduces the representation referred to in Theorem 1.2. Section 4 presents a version of Krammer's linearity proof generalized to Artin groups of arbitrary types and applies it to the representation of the preceding section. The paper finishes with a few remarks on alternative proofs in Section 5.

2. Basic properties of Artin groups

We maintain the notation of the introduction. The Coxeter group W is assumed to be finite of type A, D, or E.

The submonoid of B generated by $s_1, \ldots, s_n$ is denoted by B^+. By $\leq$ we denote the partial order on B^+ given by $x \leq y \Leftrightarrow y \in xB^+$. The length function on W with respect to R, as well as the length function on B with respect to $\{s_1, \ldots, s_n\}$, is denoted by l.

PROPOSITION 2.1: *The Artin monoid B^+ satisfies the following properties.*

(i) *The relations* (I) *form a presentation for B^+ as a monoid generated by $s_1, \ldots, s_n$.*

(ii) $B = (B^+)^{-1}B^+$. *Consequently, if* ρ *is a faithful linear representation of the monoid* B^+ *such that* $\rho(s_i)$ *is invertible, then* ρ *extends uniquely to a faithful linear representation of* B.
(iii) *For* $x, y, z \in B^+$ *we have* $zx \leq zy \Leftrightarrow x \leq y$.
(iv) *There is a uniquely determined map* $b\colon W \to B^+$ *satisfying* $b(uv) = b(u)b(v)$ *whenever* $u, v \in W$ *with* $l(uv) = l(u) + l(v)$. *It is injective and satisfies* $l(b(u)) = l(u)$. *Write* $\Omega = b(W) \subset B^+$.
(v) *There is a uniquely determined map* $L\colon B^+ \to \Omega \subseteq B^+$ *such that, for each* $x \in B^+$, $b^{-1}L(x)$ *is the longest element* w *of* W *with the property that* $b(w) \leq x$.
(vi) *For* x, $y \in B^+$, *we have* $L(xy) = L(xL(y))$.
(vii) *The map* $B^+ \times \Omega \to \Omega$ *which takes* (x, y) *to* $L(xy)$ *defines an action of* B^+ *on* Ω.

Proof: (i), the first part of (ii), (iii), and (iv) go back to [5]; they are also stated in [4, 3]. The second part of (ii) is a direct consequence of the first part (observed in [6]).

(v) See [3].

(vi) This is Corollary 1.23 of [5] (cf. Lemma 2.4 of [3]).

(vii) As observed in [6], this is immediate from (vi). ∎

A subset A of Φ^+ is called **closed** when

$$\alpha, \beta \in A,\ \alpha + \beta \in \Phi^+ \Rightarrow \alpha + \beta \in A.$$

By $\mathcal{C}$ we denote the collection of all closed subsets of Φ^+. For $w \in W$, set

$$\Phi_w = \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}.$$

Let $\mathcal{D}$ be the collection of all Φ_w ($w \in W$).

On W we have a partial order given by

$$v \leq w \Leftrightarrow \exists_{u \in W}\ w = vu \quad \text{and} \quad l(w) = l(u) + l(v). \tag{2}$$

On $\mathcal{C}$, we consider the partial order by inclusion.

LEMMA 2.2: *The members of* $\mathcal{D}$ *have the following properties.*
(i) *If* v, u, w *are as in* (2), *then* $\Phi_w = \Phi_v \cup v(\Phi_u)$.
(ii) *The size of* Φ_w *equals* $l(w)$.
(iii) *For* $x, y \in W$ *we have* $\Phi_{xy} = \Phi_x \cup x\Phi_y$ *if and only if* $\Phi_x \subseteq \Phi_{xy}$.
(iv) *The members of* $\mathcal{D}$ *are closed.*

(v) *If A is a closed subset of Φ^+, then there is a unique maximal subset A' of A of the form Φ_w with $w \in W$.*

(vi) *There is an isomorphism of partially ordered sets $(W, \leq) \to (\mathcal{D}, \subseteq)$ given by $w \mapsto \Phi_w$.*

Proof: (i) and (ii). See [2].

(iii) Clearly, one implication is trivial and the case $l(x) = 1$ follows from the fact that, for a fundamental reflection r, the set Φ_r is the singleton of the corresponding fundamental root. If $x \in W$ has length greater than one, there is a fundamental reflection r such that $u = rx$ has length $l(x) - 1$. Now $\Phi_x \subseteq \Phi_{xy}$ implies $\Phi_r \cup r\Phi_u \subseteq \Phi_r \cup r\Phi_{uy}$. Hence $\Phi_u \subseteq \Phi_{uy}$ and $\Phi_r \subseteq \Phi_{xy}$. By the induction hypothesis the former inclusion yields $\Phi_{uy} = \Phi_u \cup u\Phi_y$. But then also, by the latter inclusion and the induction hypothesis, $\Phi_{xy} = \Phi_r \cup r\Phi_{uy} = \Phi_r \cup r\Phi_u \cup ru\Phi_y = \Phi_x \cup x\Phi_y$.

(iv) For $\beta, \gamma \in \Phi_w$, we have $w^{-1}(\beta + \gamma) \in (\Phi^- + \Phi^-) \cap \Phi \subseteq \Phi^-$.

(v) For $x, y \in W$ we have $x \leq xy$ if and only if $l(xy) = l(x) + l(y)$, if and only if $\Phi_{xy} = \Phi_x \cup x(\Phi_y)$. By (iii), this is equivalent to $\Phi_x \subseteq \Phi_{xy}$.

Suppose now that there is no largest member of $\mathcal{D}$ contained in A. Then, by (i), there are $u \in W$ and $i, j \in \{1, \ldots, n\}$ with $u \leq ur_i$ and $u \leq ur_j$ for which $\Phi_{ur_i} \subseteq A$ and $\Phi_{ur_j} \subseteq A$ such that no member of $\mathcal{D}$ containing $\Phi_{ur_i} \cup \Phi_{ur_j}$ is a subset of A. Then $u\alpha_i$ and $u\alpha_j$ are in A, and, by [2], $u \leq uw_{ij}$, where w_{ij} is the longest element of the subgroup of W generated by r_i and r_j. This is $r_i r_j$ if r_i and r_j commute and $r_j r_i r_j$ if they do not. So, in the former case $\Phi_{uw_{ij}} \setminus \Phi_w$ consists of $u\alpha_i$ and $u\alpha_j$, and in the latter case of $u\alpha_i$, $u\alpha_j$, and $u\alpha_i + u\alpha_j$, which belongs to A as A is closed. This means $\Phi_{uw_{ij}} \subseteq A$, a contradiction with $\Phi_{ur_i} \cup \Phi_{ur_j} \subseteq \Phi_{uw_{ij}}$.

(vi) We have $l(w) = |\Phi_w|$ so we can work by induction on $l(w)$. Clearly, Φ_1 is the empty set, so assume $l(w) > 1$. Then there is a fundamental reflection r_i such that $l(r_i w) < l(w)$. Now $w^{-1}(\alpha_i) \in \Phi^-$ so, if $\alpha_i \in \Phi_w = \Phi_v$, then also $v^{-1}(\alpha_i) \in \Phi^-$, so $l(r_i v) < l(v)$. Consequently, $\Phi_{r_i w} = \Phi_{r_i v}$, so by induction $r_i w = r_i v$, establishing $w = v$. ∎

For A a closed subset of Φ^+, write $g(A) = x$ for $x \in \Omega$ such that $\Phi_{\delta^{-1}x}$ is the maximal subset of A belonging to $\mathcal{D}$. In view of Proposition 2.1 and Lemma 2.2, the map $g\colon \mathcal{C} \to \Omega$ is well defined.

In the next section we define the linear representation for B of type A, D, E. In the subsequent section, we use this representation to define an action of B^+ on $\mathcal{C}$ that makes the map g equivariant with the action on Ω of Proposition 2.1 (vii).

3. The representation for types A, D, E

In this section, we continue to assume that the type M is A, D, or E. This has the consequences that Φ^+ is finite and that M can be viewed as a graph on $\{1, \ldots, n\}$ with adjacency $k \sim l$ given by $(r_k r_l)^3 = 1$, or, equivalently, $(\alpha_k, \alpha_l) = -1$. Nonadjacency of k and l corresponds to $(r_k r_l)^2 = 1$ and to $(\alpha_k, \alpha_l) = 0$.

We shall first describe the '$t = 0$ part' of the linear representation of the Artin monoid B^+. Recall the $\mathbb{Z}[r^{\pm 1}, t^{\pm 1}]$-module V and the linear transformations τ_k introduced in Theorem 1.2. Denote by V_0 the free $\mathbb{Z}[r]$-module with generators x_β ($\beta \in \Phi^+$). Thus, V_0 is contained in V, and V is obtained from V_0 by extending scalars to $\mathbb{Z}[r^{\pm 1}, t^{\pm 1}]$.

LEMMA 3.1: *There is a monoid homomorphism $B^+ \to \mathrm{End}(V_0)$ determined by $s_i \mapsto \tau_i$ ($i = 1, \ldots, n$).*

Proof: We must show that, if i and j are not adjacent, then $\tau_i\tau_j = \tau_j\tau_i$ and, if they are adjacent, then $\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j$. We evaluate the expressions on each x_β and show they are equal. We begin with the case in which $\beta = \alpha_i$ or α_j. To be specific, let $\beta = \alpha_i$. Suppose first that i and j are not adjacent. Then $\tau_i x_{\alpha_i} = 0$ and $\tau_j x_{\alpha_i} = x_{\alpha_i}$. Now $\tau_j\tau_i x_{\alpha_i} = 0$, $\tau_i\tau_j x_{\alpha_i} = \tau_i x_{\alpha_i} = 0$ and the result holds. Suppose next that i and j are adjacent. Then $\tau_i x_{\alpha_i} = \tau_j x_{\alpha_j} = 0$ and $\tau_j x_{\alpha_i} = (1 - r^2)x_{\alpha_i} + r x_{\alpha_i + \alpha_j}$. Now

$$\tau_i\tau_j\tau_i x_{\alpha_i} = \tau_i\tau_j(0) = 0$$

and

$$\begin{aligned}\tau_j\tau_i\tau_j x_{\alpha_i} &= \tau_j\tau_i((1 - r^2)x_{\alpha_i} + r x_{\alpha_i+\alpha_j}) = \tau_j(0 + r^2 x_{\alpha_i+\alpha_j-\alpha_i}) \\ &= r^2 \tau_j x_{\alpha_j} = 0.\end{aligned}$$

We now divide the verifications into the various cases depending on the inner products (α_i, β) and (α_j, β). The table below describes the images of the vectors x_β under τ_i and τ_j.

(α_i, β)	(α_j, β)	$\tau_i x_\beta$	$\tau_j x_\beta$
0	0	x_β	x_β
1	1	$r x_{\beta-\alpha_i}$	$r x_{\beta-\alpha_j}$
1	-1	$r x_{\beta-\alpha_i}$	$(1 - r^2)x_\beta + r x_{\beta+\alpha_j}$
1	0	$r x_{\beta-\alpha_i}$	x_β
-1	0	$(1 - r^2)x_\beta + r x_{\beta+\alpha_i}$	x_β
-1	-1	$(1 - r^2)x_\beta + r x_{\beta+\alpha_i}$	$(1 - r^2)x_\beta + r x_{\beta+\alpha_j}$

First assume that $(\alpha_i, \alpha_j) = 0$. The computations verifying $\tau_i\tau_j = \tau_j\tau_i$ are straightforward. We summarize the results in the following table.

(α_i, β)	(α_j, β)	$\tau_i\tau_j x_\beta = \tau_j\tau_i x_\beta$
0	0	x_β
1	1	$r^2 x_{\beta-\alpha_i-\alpha_j}$
1	−1	$r(1-r^2)x_{\beta-\alpha_i} + r^2 x_{\beta+\alpha_j-\alpha_i}$
1	0	$r x_{\beta-\alpha_i}$
−1	0	$(1-r^2)x_\beta + r x_{\beta+\alpha_i}$
−1	−1	$(1-r^2)^2 x_\beta + r(1-r^2)(x_{\beta+\alpha_i} + x_{\beta+\alpha_j}) + r^2 x_{\beta+\alpha_i+\alpha_j}$

We demonstrate how to derive these expressions by checking the third line:

$$\tau_i\tau_j x_\beta = \tau_i((1-r^2)x_\beta + r x_{\beta+\alpha_j}) = (1-r^2) r x_{\beta-\alpha_i} + r^2 x_{\beta+\alpha_j-\alpha_i}.$$

In the other order,

$$\tau_j\tau_i x_\beta = \tau_j(r x_{\beta-\alpha_i}) = (1-r^2) r x_{\beta-\alpha_i} + r^2 x_{\beta+\alpha_j-\alpha_i}.$$

Recall $(\alpha_i, \alpha_j) = 0$ and so $(\beta - \alpha_i, \alpha_j) = (\beta, \alpha_j) = -1$.

Suppose then that $(\alpha_i, \alpha_j) = -1$. The same situation occurs except the computations are sometimes longer and one case does not occur. This is the case where $(\alpha_i, \beta) = (\alpha_j, \beta) = -1$. For then $\beta + \alpha_i$ is also a root, and $(\beta + \alpha_i, \alpha_j) = -1 - 1 = -2$. This means $\beta + \alpha_i = -\alpha_j$ and β is not a positive root. The table is as follows.

(α_i, β)	(α_j, β)	$\tau_i\tau_j\tau_i x_\beta = \tau_j\tau_i\tau_j x_\beta$
0	0	x_β
1	1	0
1	−1	$r(1-r^2)x_{\beta-\alpha_i} + r^2 x_\beta$
1	0	$r^2 x_{\beta-\alpha_i-\alpha_j}$
−1	0	$(1-r^2)x_\beta + (1-r^2) r x_{\beta+\alpha_i} + r^2 x_{\beta+\alpha_i+\alpha_j}$
−1	−1	does not occur

As above, these calculations are routine. Note that, in the second line, $\beta = \alpha_i + \alpha_j$. We do the second from last case in detail. Here, $(\alpha_i, \beta) = -1$ and $(\alpha_j, \beta) = 0$:

$$\begin{aligned}\tau_j\tau_i\tau_j x_\beta =& \tau_j\tau_i x_\beta = \tau_j((1-r^2)x_\beta + r x_{\beta+\alpha_i}) \\ =&(1-r^2)x_\beta + r(1-r^2)x_{\beta+\alpha_i} + r^2 x_{\beta+\alpha_i+\alpha_j}.\end{aligned}$$

In the other order,

$$\begin{aligned}\tau_i\tau_j\tau_i x_\beta =&\tau_i\tau_j((1-r^2)x_\beta + r x_{\beta+\alpha_i})\\ =&\tau_i((1-r^2)x_\beta + r(1-r^2)x_{\beta+\alpha_i} + rr x_{\beta+\alpha_i+\alpha_j})\\ =&(1-r^2)^2 x_\beta + (1-r^2) r x_{\beta+\alpha_i} + r^2(1-r^2)x_\beta + r^2 x_{\beta+\alpha_i+\alpha_j}\\ =&(1-r^2)x_\beta + r(1-r^2)x_{\beta+\alpha_i} + r^2 x_{\beta+\alpha_i+\alpha_j}. \quad \blacksquare\end{aligned}$$

We next study the possibilities for the parameters $T_{k,\beta}$ occurring in Theorem 1.2. Recall that there we defined $\sigma_k = \tau_k + tT_k$, where $T_k x_\beta = T_{k,\beta} x_{\alpha_k}$. We shall introduce $T_{k,\beta}$ as Laurent polynomials, i.e., as elements of $\mathbb{Z}[r, r^{-1}]$, but it will turn out that these actually belong to $\mathbb{Z}[r]$ (cf. Corollary 3.7).

PROPOSITION 3.2: *Set $T_{i,\alpha_i} = r^4$ for all $i \in \{1, \ldots, n\}$. For $\sigma_i \mapsto \tau_i + tT_i$ to define a linear representation of the group B on V, it is necessary and sufficient that the equations in Table 1 are satisfied for each $k, l = 1, \ldots, n$ and each $\beta \in \Phi^+$.*

Proof: The σ_k should satisfy the relations (1) for s_k. Substituting $\tau_k + tT_k$ for s_k, we find relations for the coefficients of t^i with $i = 0, 1, 2, 3$. The constant part involves only the τ_k. It follows from Lemma 3.1 that these equations are satisfied. We shall derive all of the equations of Table 1 except for (16) from the t-linear part and the remaining one from the t-quadratic part of the relations.

The coefficients of t lead to

$$(3)\qquad T_k\tau_l = T_k \quad\text{and}\quad T_l\tau_k = T_l \quad \text{if } (\alpha_k, \alpha_l) = 0,$$

$$(4)\qquad \tau_l T_k \tau_l + T_l\tau_k\tau_l + \tau_l\tau_k T_l = \tau_k T_l \tau_k + T_k\tau_l\tau_k + \tau_k\tau_l T_k \quad \text{if } (\alpha_k, \alpha_l) = -1.$$

We focus on the consequences of these equations for the $T_{k,\beta}$. Consider the case where $(\alpha_k, \alpha_l) = 0$. Then

(α_k, β)	$\tau_k x_\beta$	$T_l\tau_k = T_l$
0	x_β	no condition
1	$r x_{\beta-\alpha_k}$	$rT_{l,\beta-\alpha_k} = T_{l,\beta}$
-1	$(1-r^2)x_\beta + r x_{\beta+\alpha_k}$	$(1-r^2)T_{l,\beta} + rT_{l,\beta+\alpha_k} = T_{l,\beta}$

Both equations say the same, namely,

$$(5)\qquad T_{l,\beta} = rT_{l,\beta-\alpha_k} \text{ if } (\alpha_k, \beta) = 1 \text{ and } (\alpha_k, \alpha_l) = 0.$$

Next, we assume $(\alpha_k, \alpha_l) = -1$. A practical rule is

$$\tau_k\tau_l x_{\alpha_k} = \tau_k((1-r^2)x_{\alpha_k} + r x_{\alpha_k+\alpha_l}) = r^2 x_{\alpha_l}.$$

We distinguish cases according to the values of (α_k, β) and (α_l, β). Since each inner product for distinct roots is one of 1, 0, -1, there are six cases to consider up to symmetry (interchange of k and l). However, the case $(\alpha_k, \beta) = (\alpha_l, \beta) = -1$ does not occur. For then $(\sigma_k\beta, \alpha_l) = -2$, a contradiction with the fact that both $\sigma_k\beta$ and α_l are positive roots.

For the sake of brevity, let us denote the images of the left hand side and the right hand side of (4) on x_β by LHS and RHS, respectively.

Case $(\alpha_k, \beta) = (\alpha_l, \beta) = 1$. Then $(\sigma_k\beta, \alpha_l) = (\beta - \alpha_k, \alpha_l) = 2$, so $\beta = \alpha_k + \alpha_l$. Now

$$\begin{aligned}RHS =&\tau_k T_l \tau_k x_\beta + T_k \tau_l \tau_k x_\beta + \tau_k \tau_l T_k x_\beta\\ =&r\tau_k T_l x_{\alpha_l} + rT_k\tau_l x_{\alpha_l} + T_{k,\alpha_k+\alpha_l}\tau_k\tau_l x_{\alpha_k}\\ =&T_{l,\alpha_l} r\tau_k x_{\alpha_l} + T_{k,\alpha_k+\alpha_l} r^2 x_{\alpha_l}\\ =&T_{l,\alpha_l} r(1-r^2)x_{\alpha_l} + T_{l,\alpha_l} r^2 x_{\alpha_k+\alpha_l} + T_{k,\alpha_k+\alpha_l} r^2 x_{\alpha_l}\\ =&(T_{l,\alpha_l} r(1-r^2) + T_{k,\alpha_k+\alpha_l} r^2)x_{\alpha_l} + T_{l,\alpha_l} r^2 x_{\alpha_k+\alpha_l}.\end{aligned}$$

Comparison with the same expression but then l and k interchanged yields LHS. This leads to the following two equations:

$$\begin{aligned}T_{k,\alpha_k+\alpha_l} =&T_{l,\alpha_l}(r - r^{-1}),\\ T_{k,\alpha_k} =&T_{l,\alpha_l}.\end{aligned} \tag{6}$$

The second one, and homogeneity of the presentation relations, allow us to scale the T_i so that

$$T_{i,\alpha_i} = r^4. \tag{7}$$

Case $(\alpha_k, \beta) = (\alpha_l, \beta) = 0$. This gives

$$\begin{aligned}RHS =&\tau_k T_l x_\beta + T_k x_\beta + \tau_k\tau_l T_k x_\beta\\ =&T_{l,\beta}\tau_k x_{\alpha_l} + T_{k,\beta} x_{\alpha_k} + T_{k,\beta}\tau_k\tau_l x_{\alpha_k}\\ =&T_{l,\beta}(1-r^2)x_{\alpha_l} + T_{l,\beta} r x_{\alpha_l+\alpha_k} + T_{k,\beta} x_{\alpha_k} + T_{k,\beta} r^2 x_{\alpha_l}\\ =&(T_{k,\beta} r^2 + T_{l,\beta}(1-r^2))x_{\alpha_l} + T_{l,\beta} r x_{\alpha_l+\alpha_k} + T_{k,\beta} x_{\alpha_k}\end{aligned}$$

and LHS can be obtained from the above by interchanging the indices k and l. Comparison of each of the coefficients of x_{α_k}, $x_{\alpha_l+\alpha_k}$, x_{α_l} gives

$$T_{k,\beta} = T_{l,\beta}. \tag{8}$$

Since the other cases come down to similar computations, we only list the results.

Case $(\alpha_k,\beta)=0$, $(\alpha_l,\beta)=-1$. Then

$$T_{k,\beta+\alpha_l} = r^{-1}T_{l,\beta} - T_{k,\beta}(r^{-1}-r), \tag{9}$$

$$T_{l,\beta+\alpha_l+\alpha_k} = T_{k,\beta} - r^{-1}(1-r^2)T_{l,\beta+\alpha_l}. \tag{10}$$

Case $(\alpha_k,\beta)=0$, $(\alpha_l,\beta)=1$. Here

$$T_{k,\beta} = T_{l,\beta-\alpha_l-\alpha_k} + (r^2-1)r^{-1}T_{k,\beta-\alpha_l}, \tag{11}$$

$$T_{l,\beta} = rT_{k,\beta-\alpha_l}. \tag{12}$$

Case $(\alpha_k,\beta)=1$, $(\alpha_l,\beta)=-1$. Now

$$T_{l,\beta} = r^{-1}T_{k,\beta-\alpha_k} - (r^{-1}-r)T_{l,\beta-\alpha_k}, \tag{13}$$

$$T_{l,\beta+\alpha_l} = rT_{k,\beta}, \tag{14}$$

$$T_{k,\beta+\alpha_l} = T_{l,\beta-\alpha_k} - (r^{-1}-r)T_{k,\beta}. \tag{15}$$

Table 1. Equations for $T_{k,\beta}$

$T_{k,\beta}$	condition	reference
0	$\beta=\alpha_l$ and $k\neq l$	(16)
r^4	$\beta=\alpha_k$	(7)
r^5-r^3	$\beta=\alpha_k+\alpha_l$	(6)
$rT_{k,\beta-\alpha_l}$	$(\alpha_l,\beta)=1$ and $(\alpha_k,\alpha_l)=0$	(5)
$T_{l,\beta-\alpha_k-\alpha_l}+(r-r^{-1})T_{k,\beta-\alpha_l}$	$(\alpha_k,\beta)=0$ and $(\alpha_l,\beta)=1$ and $(\alpha_k,\alpha_l)=-1$	(11)
$r^{-1}T_{l,\beta-\alpha_l}+(r-r^{-1})T_{k,\beta-\alpha_l}$	$(\alpha_k,\beta)=-1$ and $(\alpha_l,\beta)=1$ and $(\alpha_k,\alpha_l)=-1$	(13)
$rT_{l,\beta-\alpha_k}$	$(\alpha_k,\beta)=1$ and $(\alpha_l,\beta)=0$ and $(\alpha_k,\alpha_l)=-1$	(12)

We see that, in order to be a representation, the $T_{i,\beta}$ have to satisfy the equations (5)–(15). In the t-quadratic study below, we shall also derive the equation (16). The resulting system (5)–(16) is superfluous in that, when the root in the index of the left hand side of (9) is set to γ, we obtain (13) for γ instead of β. Similarly, (10) is equivalent to (11), while (14) is equivalent to (12) and (15) is equivalent to (11).

We also contend that the equations in (8) are consequences of the other relations from Table 1. The equation says that $T_{k,\beta}=T_{l,\beta}$ whenever $(\alpha_k,\beta)=(\alpha_l,\beta)=0$ and $k\sim l$. We prove this by induction on the height of β. The initial case of β having height 1 is direct from (16). Suppose therefore $\mathrm{ht}(\beta) > 1$. There exists $m \in \{1, \dots, n\}$ such that $(\alpha_m, \beta) = 1$. If $(\alpha_m, \alpha_k) = (\alpha_m, \alpha_l) = 0$, then (5) applies to both sides, giving $T_{k,\beta} = rT_{k,\beta-\alpha_m} = rT_{l,\beta-\alpha_m} = T_{l,\beta}$, where the middle step uses the induction hypothesis.

Therefore, interchanging k and l if necessary, we may assume that $l \sim m$, whence $k \not\sim m$ (as the Dynkin diagram contains no triangles). Now $\delta = \beta - \alpha_m - \alpha_l \in \Phi^+$ and $(\alpha_k, \delta) = 1$, so (5) gives $T_{m,\delta} = rT_{m,\delta-\alpha_k}$, which, by induction on height, is equal to $rT_{l,\delta-\alpha_k}$ (as $(\alpha_l, \delta - \alpha_k) = (\alpha_m, \delta - \alpha_k) = 0$). Notice that β has height at least 3. Consequently,

$$\begin{aligned} T_{k,\beta} =& rT_{k,\beta-\alpha_m} && \text{by (5)} \\ =& rT_{l,\delta-\alpha_k} + (r^2-1)T_{k,\delta} && \text{by (11)} \\ =& T_{m,\delta} + (r^2-1)T_{k,\delta} && \text{by the above} \\ =& T_{m,\delta} + (r - r^{-1})T_{l,\beta-\alpha_m} && \text{by (12)} \\ =& T_{l,\beta} && \text{by (11).} \end{aligned}$$

We have established that the equations of Table 1 represent a system of equations equivalent to (3), (4), and (16).

We now consider the coefficients of t^2 and of t^3 in the equations (1) for σ_i. We claim that, given (5)–(15), a necessary condition for the corresponding equations to hold is

$$T_{k,\alpha_l} = 0 \quad \text{if } k \neq l. \tag{16}$$

To see this, note that, if $k \not\sim l$, the coefficient of t^2 gives $T_k T_l = T_l T_k$ which, applied to x_{α_l}, yields (16). If $k \sim l$, note

$$T_k \tau_l x_{\alpha_k} = T_k((1-r^2)x_{\alpha_k} + r x_{\alpha_k+\alpha_l}) = (r^4(1-r^2) + rT_{k,\alpha_k+\alpha_l})x_{\alpha_k} = 0$$

as $T_{k,\alpha_k+\alpha_l} = r^5 - r^3$. Now use the action of

$$T_l \tau_k T_l + \tau_l T_k T_l + T_l T_k \tau_l = T_k \tau_l T_k + \tau_k T_l T_k + T_k T_l \tau_k$$

on x_{α_l}. We see only the middle terms do not vanish because of the relation above and so

$$r^4 T_{k,\alpha_l} \tau_l x_{\alpha_k} = T_{k,\alpha_l} T_{l,\alpha_k} \tau_k x_{\alpha_l}.$$

By considering the coefficient of x_{α_k}, which occurs only on the left hand side, we see that (16) holds.

A consequence of this is that $T_i T_j = 0$ if $i \neq j$. Now all the equations for the t^2 and t^3 coefficients are easily satisfied. In the noncommuting case of t^2, the first terms on either side are 0 by the relation above and the other terms are 0 as $T_l T_k = 0$.

It remains to establish that the matrices σ_k are invertible. To prove this, we observe that the linear transformation $\sigma_k^2 + (r^2 - 1)\sigma_k - r^2$ maps V onto the submodule spanned by x_{α_k} and that the image of x_{α_k} under σ_k is $tr^4 x_{\alpha_k}$. In fact, the determinant of σ_k equals $(-1)^c tr^{4+2c}$, where c is the number of positive roots β such that $(\alpha_k, \beta) = -1$. ∎

For a positive root β, we write $\mathrm{ht}(\beta)$ to denote its height, that is, the sum of its coefficients with respect to the α_i. Also, $\mathrm{Supp}(\beta)$ is the set of $k \in \{1, \ldots, n\}$ such that the coefficient of α_k in β is nonzero.

COROLLARY 3.3: *If the $T_{k,\beta} \in \mathbb{Z}[r, r^{-1}]$ satisfy the equations in Table 1, then these obey the following rules, where $\mathrm{ht}(\beta)$ stands for the height of β.*

(i) *If $(\alpha_k, \beta) = (\alpha_l, \beta) = 0$ and $(\alpha_k, \alpha_l) = -1$, then $T_{k,\beta} = T_{l,\beta}$.*
(ii) *If $(\alpha_k, \beta) = 1$, then $T_{k,\beta} = r^{\mathrm{ht}(\beta)+1}(r^2 - 1)$.*
(iii) *The degree of $T_{k,\beta}$ equals $3 + \mathrm{ht}(\beta)$ whenever $k \in \mathrm{Supp}(\beta)$.*
(iv) *$T_{k,\beta}$ is a multiple of $r^2 - 1$ whenever $\beta \neq \alpha_k$.*
(v) *$T_{k,\beta} = 0$ whenever $k \notin \mathrm{Supp}(\beta)$.*

Proof: (i) The equations are necessary as they appeared under (8).

(ii) Use induction on $\mathrm{ht}(\beta)$. If $\mathrm{ht}(\beta) = 2$, the equation coincides with (6). If $\mathrm{ht}(\beta) > 2$, then either (5) or (12) applies. As $\mathrm{ht}(\beta) \geq 2$ there must be some l for which $(\beta - \alpha_k, \alpha_l) = 1$. Now $(\beta, \alpha_l) - (\alpha_k, \alpha_l) = 1$. If $(\alpha_k, \alpha_l) = 0$, then $(\beta, \alpha_l) = 1$ and (5) applies; if $(\alpha_k, \alpha_l) = -1$, then $(\beta, \alpha_l) = 0$ and (12) applies.

(iii) and (iv) are obvious.

(v) follows from (16) by use of (5) and (13). Observe that, if $k \notin \mathrm{Supp}(\beta)$ and $(\alpha_l, \beta) = 1$ for some $l \sim k$, then $l \notin \mathrm{Supp}(\beta - \alpha_l)$. ∎

The proposition enables us to describe an algorithm computing the $T_{k,\beta}$, and which shows that there is at most one solution.

Algorithm 3.4: The Laurent polynomials $T_{k,\beta}$ of Theorem 1.2 can be computed as follows by using Table 1.

(i) If $k \notin \mathrm{Supp}(\beta)$, then $T_{k,\beta} = 0$. Otherwise, proceed with the next steps.
(ii) If $\mathrm{ht}(\beta) \leq 2$, equations (7) and (6), that is, the second and third lines of Table 1, determine $T_{k,\beta}$. From now on, assume $\mathrm{ht}(\beta) > 2$. We proceed by recursion, expressing $T_{k,\beta}$ in $\mathbb{Z}[r, r^{-1}]$-linear combinations of $T_{m,\gamma}$'s with $\mathrm{ht}(\gamma) < \mathrm{ht}(\beta)$.

(iii) If $(\alpha_k, \beta) = 1$, set $T_{k,\beta} = r^{\mathrm{ht}(\beta)+1}(r^2 - 1)$. From now on, assume (α_k, β) is 0 or -1.

(iv) Search for an $l \in \{1, \ldots, n\}$ such that $(\alpha_k, \alpha_l) = 0$ (so k and l are nonadjacent in M) and $(\alpha_l, \beta) = 1$ (so $\beta - \alpha_l \in \Phi$). If such an l exists, then (5) expresses $T_{k,\beta}$ as a multiple of $T_{k,\beta-\alpha_l}$.

(v) So, suppose there is no such l. We know $\beta - \alpha_k$ is not a root as $(\alpha_k, \beta) \neq 1$. There is an l for which $\beta - \alpha_l$ is a root (so $(\alpha_l, \beta) = 1$). We must have $(\alpha_k, \alpha_l) = -1$ (so k and l are adjacent in M). By our choice, $(\alpha_k, \beta) = 0$ or -1. Now the identities (11) or (13) express $T_{k,\beta}$ as a linear combination of $T_{k,\beta-\alpha_l}$ and some $T_{l,\gamma}$ with $\mathrm{ht}(\gamma) < \mathrm{ht}(\beta)$.

This ends the algorithm. Observe that all lines of Table 1 have been used (with (16) in (i) and (12) implicitly in (iv)).

The algorithm computes a Laurent polynomial for each k, β based on Table 1, showing that there is at most one solution to the set of equations. The next result shows that the computed Laurent polynomials do indeed give a solution.

PROPOSITION 3.5: *The equations of Table 1 have a unique solution.*

Proof: We need to show that the Laurent polynomials $T_{k,\beta}$ defined by Algorithm 3.4 satisfy the equations of Table 1. By Step (i), (16) is satisfied. By Step (ii), (7) and (6) are satisfied if β has height 1 or 2. We use induction on $\mathrm{ht}(\beta)$, the height of β, and assume $\mathrm{ht}(\beta) \geq 3$. Suppose first $(\alpha_k, \beta) = 1$. Notice in (5) that $(\beta, \beta - \alpha_l) = 1$ as $(\alpha_l, \beta) = 1$ and in (12) that $(\beta, \beta - \alpha_k) = 1$ as here $(\beta, \alpha_k) = 1$. This means the relevant terms are defined by Step (iii) which depends only on the heights. As $\mathrm{ht}(\beta) = \mathrm{ht}(\beta - \alpha_j) + 1$ the equations are correct: here $T_{k,\beta} = r^{\mathrm{ht}(\beta)+1}(r^2 - 1)$, so if $(\alpha_l, \beta) = 1$, we have $T_{k,\beta-\alpha_l} = r^{\mathrm{ht}(\beta-\alpha_l)+1}(r^2 - 1)$ whence $T_{k,\beta} = rT_{k,\beta-\alpha_l}$ giving (5). The proof of (12) is similar. This shows (12) holds and (5) holds if $(\alpha_k, \beta) = 1$.

We now suppose that (α_k, β) is 0 or -1. We first check (5). If this applies, the value $T_{k,\beta}$ is determined in Step (iv) of the algorithm, and we are really checking the value did not depend on the choice of l. Suppose there is an l' for which $(\alpha_l, \beta) = (\alpha_{l'}, \beta) = 1$ and $(\alpha_l, \alpha_k) = (\alpha_{l'}, \alpha_k) = 0$. Then by our definition $T_{k,\beta} = rT_{k,\beta-\alpha_{l'}}$ and we must show that $T_{k,\beta} = rT_{k,\beta-\alpha_l}$. If $l \sim l'$, then $(\beta - \alpha_k, \alpha_l) = 2$ and $\beta = \alpha_k + \alpha_l$ has height 2. This means we can assume $l \not\sim l'$. Then $(\beta - \alpha_l, \alpha_{l'}) = 1$ and $(\beta - \alpha_{l'}, \alpha_l) = 1$. In particular, $\beta - \alpha_l - \alpha_{l'}$ is also a root. Now apply (5) and the induction hypothesis to see $T_{k,\beta-\alpha_l} = rT_{k,\beta-\alpha_l-\alpha_{l'}}$ and $T_{k,\beta-\alpha_{l'}} = rT_{\beta-\alpha_l-\alpha_{l'}}$, and so $T_{k,\beta} = rT_{k,\beta-\alpha_l} = rT_{k,\beta-\alpha_{l'}}$. This shows that (5) holds.

We have yet to check (11) and (13). Suppose first $T_{k,\beta}$ was chosen by Step (iv). In this case there are l, l' with $(\beta,\alpha_l)=(\beta,\alpha_{l'})=1$, $(\alpha_k,\alpha_l)=-1$ and $(\alpha_k,\alpha_{l'})=0$. Here $T_{k,\beta}$ is determined by Step (iv) of the algorithm, $T_{k,\beta}=rT_{k,\beta-\alpha_{l'}}$. We must have $l\not\sim l'$, for if $l\sim l'$, we would again be in the height 2 case. In order to obtain (11) we must show that if $(\alpha_k,b)=0$, then

$$rT_{k,\beta-\alpha_{l'}}=T_{l,\beta-\alpha_k-\alpha_l}+(r-r^{-1})T_{k,\beta-\alpha_l}.$$

Observe that $(\beta-\alpha_{l'},\alpha_l)=1$ and $(\alpha_k,\alpha_l)=-1$ and so, by (11),

$$rT_{k,\beta-\alpha_{l'}}=rT_{l,\beta-\alpha_{l'}-\alpha_l-\alpha_k}+r(r-r^{-1})T_{k,\beta-\alpha_{l'}-\alpha_l}.$$

Now, as $(\alpha_l,\alpha_{l'})=0$, we can use (5) to obtain $T_{l,\beta-\alpha_k-\alpha_l}=rT_{l,\beta-\alpha_l-\alpha_k-\alpha_{l'}}$ and $T_{k,\beta-\alpha_l}=rT_{k,\beta-\alpha_l-\alpha_{l'}}$, and so the equations are satisfied. In order to satisfy (13) when $(\beta,\alpha_k)=-1$, we need to show

$$rT_{k,\beta-\alpha_{l'}}=r^{-1}T_{l,\beta-\alpha_l}+(r-r^{-1})T_{k,\beta-\alpha_l}.$$

Again express these terms using (5) subtracting $\alpha_{l'}$ in each of the expressions to get equality.

We may now assume that $T_{k,\beta}$ was chosen in Step (v). If l is the one chosen in Step (v), then $T_{k,\beta}$ was chosen to satisfy (11) or (13), whichever it is. If not, there is another index l' which was used in Step (v) to define $T_{k,\beta}$. For these the conditions are $(\beta,\alpha_l)=(\beta,\alpha_{l'})=1$ and $(\alpha_k,\alpha_l)=(\alpha_k,\alpha_{l'})=-1$. Clearly $l\not\sim l'$, for otherwise there would be a triangle in the Dynkin diagram M. Now $(\alpha_{l'},\beta-\alpha_k-\alpha_l)=2$ and so β has height 3. Now the necessary conditions follow as the terms are of height 1 or 2 in the expression

$$T_{l,\beta-\alpha_k-\alpha_l}+(r+r^{-1})T_{k,\beta-\alpha_l}=T_{l',\beta-\alpha_k-\alpha_{l'}}+(r+r^{-1})T_{k,\beta-\alpha_{l'}}$$

for (11) and of height 2 for (13).

Now all the equations in Table 1 have been shown to hold. ∎

COROLLARY 3.6: *The solution $T_{k,\beta}$ of Proposition 3.5 is computable via exponents $a_{k,\beta}$, $c_{k,\beta}$, $d_{k,\beta}$ as follows. $T_{k,\beta}=0$ if $k\notin\mathrm{Supp}(\beta)$ which amounts to $a_{k,\beta}$, $c_{k,\beta}$, $d_{k,\beta}$ being zero. Moreover, $T_{k,\beta}=r^4$ if $\beta=\alpha_k$. Otherwise,*

$$\frac{T_{k,\beta}}{r^{\mathrm{ht}(\beta)+1}(r^2-1)}=\begin{cases}1 & \text{if } (\alpha_k,\beta)=1\\ (1-r^{-a_{k,\beta}}) & \text{if } (\alpha_k,\beta)=0\\ (1-r^{-c_{k,\beta}})(1-r^{-d_{k,\beta}}) & \text{if } (\alpha_k,\beta)=-1\end{cases}\tag{17}$$

with

$$\begin{aligned}
a_{k,\beta} =& a_{k,\beta-\alpha_l} && \textit{if } (\alpha_l,\beta)=1 \textit{ and } k \not\sim l,\\
a_{k,\beta} =& a_{l,\beta-\alpha_l-\alpha_k}+2 && \textit{if } (\alpha_l,\beta)=1 \textit{ and } k \sim l,\\
\{c_{k,\beta},d_{k,\beta}\} =& \{c_{k,\beta-\alpha_l},d_{k,\beta-\alpha_l}\} && \textit{if } (\alpha_l,\beta)=1 \textit{ and } k \not\sim l,\\
\{c_{k,\beta},d_{k,\beta}\} =& \{a_{k,\beta-\alpha_l},c_{l,\beta-\alpha_l}+2\} && \textit{if } (\alpha_l,\beta)=1,\ d_{l,\beta-\alpha_l}=a_{k,\beta-\alpha_l},\\
& && \textit{and } k\sim l\\
\{c_{k,\beta},d_{k,\beta}\} =& \{a_{l,\beta},a_{m,\beta}\} && \textit{if } (\alpha_l,\beta)=0,\ (\alpha_m,\beta)=0,\\
& && \textit{and } l\sim k\sim m\neq l.
\end{aligned}$$

Proof: The proof is similar to that of Proposition 3.2. It runs by induction on the height of β. The initial cases and the case where $(\alpha_k,\beta)=1$ follow directly from Proposition 3.2 and Corollary 3.3.

Suppose $(\alpha_k,\beta)=0$. Let l be such that $(\alpha_l,\beta)=1$. If $l\not\sim k$, then (5) applies, which, in view of the induction hypothesis and $(\alpha_k,\beta-\alpha_l)=0$, gives

$$T_{k,\beta}=rT_{k,\beta-\alpha_l}=r^{\mathrm{ht}(\beta)+1}(r^2-1)(1-r^{-a_{k,\beta-\alpha_l}}),$$

proving the first rule. If $l\sim k$, then (11) applies, which, in view of the induction hypothesis and $(\alpha_k,\beta-\alpha_l)=1$ and $(\alpha_l,\beta-\alpha_k-\alpha_l)=0$, gives

$$\begin{aligned}
T_{k,\beta} =& T_{l,\beta-\alpha_k-\alpha_l}+(r-r^{-1})T_{k,\beta-\alpha_l}\\
=& r^{\mathrm{ht}(\beta)-1}(r^2-1)(1-r^{-a_{l,\beta-\alpha_k-\alpha_l}})+r^{\mathrm{ht}(\beta)}(r^2-1)(r-r^{-1})\\
=& r^{\mathrm{ht}(\beta)-1}(r^2-1)(1-r^{-a_{l,\beta-\alpha_k-\alpha_l}}+r^2-1)\\
=& r^{\mathrm{ht}(\beta)+1}(r^2-1)(1-r^{-a_{l,\beta-\alpha_k-\alpha_l}-2}),
\end{aligned}$$

proving the second rule.

Next suppose $(\alpha_k,\beta)=-1$. Let l be such that $(\alpha_l,\beta)=1$. If $k\not\sim l$, then the third rule follows from (5).

If $k\sim l$, then $(\alpha_l,\beta-\alpha_l)=-1$ and $(\alpha_k,\beta-\alpha_l)=0$, so (13), induction and $a_{k,\beta-\alpha_l}=d_{l,\beta-\alpha_l}$ give

$$\begin{aligned}
T_{k,\beta} =& r^{-1}T_{l,\beta-\alpha_l}+(r-r^{-1})T_{k,\beta-\alpha_l}\\
=& r^{\mathrm{ht}(\beta)+1}(r^2-1)(1-r^{-a_{k,\beta-\alpha_l}})\left((1-r^{-c_{l,\beta-\alpha_l}})r^{-2}+(1-r^{-2})\right)\\
=& r^{\mathrm{ht}(\beta)+1}(r^2-1)(1-r^{-a_{k,\beta-\alpha_l}})(1-r^{-c_{l,\beta-\alpha_l}-2}),
\end{aligned}$$

as required for the fourth rule.

In order to prove the last rule of the corollary, suppose $(\alpha_k,\beta)=-1$ and let $l,m\in\mathrm{Supp}(\beta)$ be as indicated. Let j be such that $(\alpha_j,\beta)=1$. If j is nonadjacent to each of k, l, m, then the rule follows easily from (5). Therefore, we may and shall assume that each index j with $(\alpha_j, \beta) = 1$ is adjacent to (at least and hence exactly) one of k, l, m. By analysis of the root system of type M, it follows that we can always choose j to be adjacent with either l or m. Thus, without loss of generality, assume that there exists an index j with $(\alpha_j, \beta) = 1$ and $j \sim m$. Then $j \not\sim k, l$ as the Coxeter graph is a tree. According to (5), (13), and the induction hypothesis, we have

$$\begin{aligned}
T_{k,\beta} =& rT_{k,\beta-\alpha_j} = T_{m,\beta-\alpha_j-\alpha_m} + (r^2-1)T_{k,\beta-\alpha_j-\alpha_m} \\
=& r^{\mathrm{ht}(\beta)-1}(r^2-1)(1-r^{-a_{j,\beta-\alpha_j-\alpha_m}})(1-r^{-a_{k,\beta-\alpha_j-\alpha_m}}) \\
&+ r^{\mathrm{ht}(\beta)-1}(r^2-1)(1-r^{-a_{k,\beta-\alpha_j-\alpha_m}})(r^2-1) \\
=& r^{\mathrm{ht}(\beta)+1}(r^2-1)(1-r^{-a_{j,\beta-\alpha_j-\alpha_m}-2})(1-r^{-a_{k,\beta-\alpha_j-\alpha_m}}).
\end{aligned}$$

By the second rule $a_{j,\beta-\alpha_j-\alpha_m} + 2 = a_{m,\beta}$, and by Corollary 3.3(i) and the first rule, $a_{k,\beta-\alpha_j-\alpha_m} = a_{l,\beta-\alpha_j-\alpha_m} = a_{l,\beta-\alpha_j} = a_{l,\beta}$, whence the last rule. ∎

We are now ready to prove the first part of Theorem 1.2.

Corollary 3.7: *The Laurent polynomials $T_{k,\beta}$ of Proposition 3.5 belong to $r\mathbb{Z}[r]$. In particular, they are polynomials, and the $T_{k,\beta}$ determine a representation of B on V as claimed in Theorem 1.2.*

Proof: By induction on $\mathrm{ht}(\beta)$, the rules for $a_{k,\beta}$, $c_{k,\beta}$, $d_{k,\beta}$ of Corollary 3.6 readily imply that $a_{k,\beta} \leq \mathrm{ht}(b)$ and $c_{k,\beta} + d_{k,\beta} \leq \mathrm{ht}(b)$. Hence the first part of the corollary. For the second part, combine the above with Propositions 3.2 and 3.5. ∎

Example 3.8: The A_n case. Then $c_{k,\beta} = d_{k,\beta} = 0$ and $a_{k,\beta} = 2$ if $k \in \mathrm{Supp}(\beta)$. Note that the last three lines of the corollary do not occur. Our representation can be obtained from the Lawrence Krammer representation as described in [6] by a diagonal transformation with respect to the basis x_β ($\beta \in \Phi^+$), and by replacing q by r^2. As a result, the involutory automorphism of the diagram A_n can be realized as a linear transformation leaving invariant the basis (compare with Remark 5.1 of [6]). To be more specific, the roots in the A_n case are of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $1 \leq i < j \leq n$. For such a root β, set $x_{i,j} = (r^{-1})^{i+j} x_\beta$. These are the elements appearing in [6]. In the action of σ_k on this basis r always appears to an even power. Replacing r^2 by q gives the action in [6].

Example 3.9: The D_n case. For a given root in D_n, let l_1 be the number of coefficients 1 in the expression of β as a linear combination of the α_i, and let l_2 be the number of coefficients 2. These are the only nonzero coefficients which can occur for D_n. In the case $(\alpha_k, \beta) = -1$, we have $\{c_{k,\beta}, d_{k,\beta}\} = \{2, 2l_2 + 2\}$. Assume now that $(\alpha_k, \beta) = 0$ and $k \in \mathrm{Supp}(\beta)$. If α_k has coefficient 2 in β or k is the end node of a short branch of the Coxeter diagram, then $a_{k,\beta} = 4$; if k is the end node of the long branch (possibly after removing nodes with zero coefficients), then $a_{k,\beta} = 2l_2 + 2$; otherwise $a_{k,\beta} = 2$. It is straightforward to check that the relations of Table 1 all hold.

4. Faithfulness of the representation

We now combine the representation of Section 3 with the root system knowledge of Section 2. Our arguments are straightforward generalizations of Krammer's method, but we give details anyway for the reader's convenience.

Recall that V is the free module over $\mathbb{Z}[t^{\pm 1}, r^{\pm 1}]$ generated by x_β for β ranging over the positive roots. In Corollary 3.7 we established the first part of Theorem 1.2. In this section we prove the second part. To this end, we specialize r to a real number r_0 with $0 < r_0 < 1$ in $V \otimes \mathbb{R}$ to obtain V_1, the free module over $\mathbb{R}[t, t^{-1}]$ generated by the x_β. We also keep the Coxeter matrix M to be one of A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, or E_8.

Note that $0 < r_0 < 1$ implies that the constant term of each of the entries of the matrices σ_i is a nonnegative real number. This will be the same for any product of σ_i, and so for any element of the monoid B^+ they generate. Therefore, in its linear action on V_1, the monoid B^+ preserves

$$U = \bigoplus_{\beta \in \Phi^+} (\mathbb{R}_{\geq 0} \oplus t\mathbb{R}[t])x_\beta. \tag{18}$$

For $A \subseteq \Phi^+$ set

$$U_A = \left\{ \sum_{\beta \in \Phi^+} k_\beta x_\beta \in U \,\middle|\, k_\beta \in t\mathbb{R}[t] \Leftrightarrow \beta \in A \right\}.$$

Then, obviously, U is the disjoint union of the U_A.

LEMMA 4.1: *For $x \in B^+$ and $A \subseteq \Phi^+$, there is a unique $A' \subseteq \Phi^+$ such that $xU_A \subseteq U_{A'}$.*

Proof: For a given subset A of Φ^+, the elements of U_A are the vectors in U for which the support mod t is exactly $\Phi^+ \setminus A$. In particular, an element

$u = \sum_{\beta\in\Phi^+}(k_\beta + tp_\beta)x_\beta$ of U, with $k_\beta \in \mathbb{R}_{\geq 0}$ and $p_\beta \in \mathbb{R}[t]$, is in U_A if and only if $k_\beta = 0$ for $\beta \in A$ and $k_\beta \neq 0$ for $\beta \in \Phi^+ \setminus A$. As all matrix entries of an element x of B^+ are nonnegative mod t, the image by x acting on two nonzero elements of U_A will have exactly the same support mod t. If this is $\Phi^+ \setminus A'$, the images of nonzero vectors of U_A are all in $U_{A'}$. ∎

The assignment $(x, A) \mapsto A'$, where A' is the unique subset of Φ^+ such that $U_{A'}$ contains xU_A, defines an action of B^+ on $\mathcal{P}(\Phi^+)$; we write $x * A$ for A'. Observe that $A \subseteq D$ implies that $x * A \subseteq x * D$.

LEMMA 4.2: *The action $*$ preserves $\mathcal{C}$. It can be explicitly described for s_k as follows, where $k \in \{1, \ldots, n\}$ and $A \in \mathcal{C}$:*

$$s_k * A = \{\alpha_k\} \cup \left\{ \beta \in \Phi^+ \,\middle|\, \begin{array}{ll} \beta - \alpha_k \in A & \text{if } (\alpha_k, \beta) = 1, \\ \beta \in A & \text{if } (\alpha_k, \beta) = 0, \\ \beta, \beta + \alpha_k \in A & \text{if } (\alpha_k, \beta) = -1. \end{array} \right\}$$

In particular, $\alpha_k \in s_k * A \subseteq \{\alpha_k\} \cup r_k(A)$.

Proof: For the proof of the first statement, it suffices to consider $x = s_k$ as B^+ is generated by these elements.

As for the description of $s_k * A$, only the action of τ_k on $u = \sum_{\beta\in\Phi^+} k_\beta x_\beta \in U$ with $k_\beta \in \mathbb{R}_{\geq 0}$ is relevant. A computation shows

$$\begin{aligned} \tau_k u &= \sum_{(\gamma,\alpha_k)=-1} k_\gamma((1 - r_0^2)x_\gamma + r_0 x_{\gamma+\alpha_k}) + \sum_{(\gamma,\alpha_k)=0} k_\gamma x_\gamma + \sum_{(\gamma,\alpha_k)=1} k_\gamma r_0 x_{\gamma-\alpha_k} \\ &= \sum_{(\beta,\alpha_k)=1} k_{\beta-\alpha_k} r_0 x_\beta + \sum_{(\beta,\alpha_k)=0} k_\beta x_\beta + \sum_{(\beta,\alpha_k)=-1} (k_{\beta+\alpha_k} r_0 + k_\beta(1 - r_0^2))x_\beta. \end{aligned}$$

The set $s_k * A$ is the set of positive roots for which x_β has coefficient 0 in $\tau_k u$ for any element u in U_A. The description of $s_k * A$ follows directly from this formula. For instance, for $\beta \in \Phi^+$ with $(\beta, \alpha_k) = -1$ to belong to $s_k * A$, we need to have $k_{\beta+\alpha_k} r_0 + k_\beta(1 - r_0^2) = 0$, which is equivalent to $k_{\beta+\alpha_k} = k_\beta = 0$, whence $\beta + \alpha_k, \beta \in A$.

It remains to show that $s_k * A$ is closed. So suppose that β and γ are in $s_k * A$ and that $\beta + \gamma$ is in Φ^+. Assume $\gamma = \alpha_k$. We always have α_k in $s_k * A$. As $\beta + \alpha_k \in \Phi^+$, the inner product (α_k, β) equals -1. By the above, this implies that both β and $\beta + \alpha_k$ are in A. But then $\beta + \alpha_k \in \Phi^+$ satisfies $(\alpha_k, \beta + \alpha_k) = 1$ and $(\beta + \alpha_k) - \alpha_k \in A$, so $\beta + \alpha_k \in s_k * A$.

From now on, we assume that neither β nor γ is equal to α_k. Suppose that both β and γ are orthogonal to α_k. We saw above that being in $s_k * A$ means that both β and γ are in A and, because A is closed, $\beta+\gamma$ is also in A. But then $\beta+\gamma$, being orthogonal to α_k, also belongs to $s_k * A$.

The case remains where at least one of β and γ is not orthogonal to α_k. Suppose first that $(\alpha_k, \beta) = -1$. As $\beta \in s_k * A$, by the above, both β and $\beta+\alpha_k$ are in A. If γ is orthogonal to α_k, we know from above and from $\gamma \in s_k * A$ that γ is in $s_k * A$. Now, as $\beta+\alpha_k, \gamma \in A$ and A is closed, also $\gamma+\beta+\alpha_k \in A$. As $\beta, \gamma \in A$ and A is closed, also $\beta+\gamma$ is in A. Now $(\beta+\gamma, \alpha_k) = -1$, and so by the above $\beta+\gamma$ is in A. We still need to consider the other possibilities for (γ, α_k). As $\beta+\gamma$ is a root, $(\gamma, \alpha_k) \neq -1$. Now $(\beta+\gamma, \alpha_k) = 0$ and we need only show that $\beta+\gamma \in A$. But this follows as A is closed and $\beta+\alpha_k, \gamma-\alpha_k \in A$.

The only case remaining is $(\beta, \alpha_k) = 1$ and $(\gamma, \alpha_k) \in \{0,1\}$. However, the latter inner product cannot be 1, for otherwise $(\beta+\gamma, \alpha_k) = 2$, contradicting the fact that $\beta+\gamma$ is a positive root. This means $(\gamma, \alpha_k) = 0$ and, as $\gamma \in s_k * A$, we find $\gamma \in A$. As $(\beta, \alpha_k) = 1$ and $\beta \in s_k * A$, we have $\beta - \alpha_k \in A$. Since $(\beta+\gamma, \alpha_k) = 1$, the vector $\beta+\gamma-\alpha_k$ is a positive root. As both $\beta-\alpha_k$ and γ are in A and A is closed, the root $\beta-\alpha_k+\gamma$ belongs to A. Now as $(\beta+\gamma, \alpha_k) = 1$ and $\beta+\gamma-\alpha_k \in A$, we conclude $\beta+\gamma \in s_k * A$. ∎

LEMMA 4.3: *For $w \in W$ and $i \in \{1, \ldots, n\}$ satisfying $l(r_i w) < l(w)$, and for each closed subset A of Φ^+, we have $\Phi_w \subseteq \{\alpha_i\} \cup r_i(A)$ if and only if $w \leq b^{-1}(L(s_i g(A)))$.*

Proof: Since $l(r_i w) < l(w)$, the subset Φ_w of Φ^+ coincides with $\{\alpha_i\} \cup r_i(\Phi_{r_i w})$. Hence $\Phi_w \subseteq \{\alpha_i\} \cup r_i(A)$ if and only if $\Phi_{r_i w} \subseteq A$, which, by definition of g, is equivalent to $b(r_i w) \leq g(A)$. By Proposition 2.1(iii), this is the same as $s_i b(r_i w) \leq s_i g(A)$, while, since the left hand side equals $b(w)$, this in turn amounts to $b(w) \leq L(s_i g(A))$. Hence the lemma. ∎

LEMMA 4.4: *Suppose that the subsets A and E of Φ^+ are closed and, for some $i \in \{1, \ldots, n\}$, satisfy $\{\alpha_i\} \subseteq E \subseteq \{\alpha_i\} \cup r_i(A)$. Then $E \subseteq s_i * A$.*

Proof: Let $\beta \in E$. We show that $\beta \in s_i * A$. We distinguish cases according to (α_i, β). If $(\alpha_i, \beta) = 2$, then $\beta = \alpha_i \in s_i * A$ by Lemma 4.2.

If $(\alpha_i, \beta) = 1$, then $\beta = r_i(\beta - \alpha_i)$ with $\beta - \alpha_i \in A$. By Lemma 4.2, this implies $\beta \in s_i * A$.

If $(\alpha_i, \beta) = 0$, then $\beta = r_i(\beta)$ with $\beta \in A$. By Lemma 4.2, this implies $\beta \in s_i * A$.

Finally, suppose $(\alpha_i, \beta) = -1$. Then $\beta = r_i(\beta + \alpha_i)$ with $\beta + \alpha_i \in A$. Notice $\beta + \alpha_i \in E$ and $E \subseteq \{\alpha_i\} \cup r_i(A)$ imply $\beta + \alpha_i = r_i(\beta) \in r_i(A)$. In particular,

$\beta + \alpha_i \in r_i(A)$, so $\beta + \alpha_i = r_i(\beta)$ with $\beta \in A$. Since $\beta, \beta + \alpha_i \in A$, Lemma 4.2 implies $\beta \in s_i * A$. Hence the lemma. ∎

PROPOSITION 4.5: *The map $g: \mathcal{C} \to \Omega$ is B^+ equivariant. That is, for all $x \in B^+$ and $A \in \mathcal{C}$, we have*

$$g(x * A) = L(xg(A)).$$

Proof: It suffices to prove the assertion for $x = s_i$ with $1 \le i \le n$. Write $w = b^{-1}g(s_i * A)$, so $b(w) = g(s_i * A)$, and Φ_w is the maximal subset of $s_i * A$ belonging to $\mathcal{D}$. Recall from Lemma 4.2 that $\alpha_i \in s_i * A$. It implies $\Phi_{r_i} \subseteq s_i * A$, whence $\Phi_{r_i} \subseteq \Phi_w$, so $r_i \le w$. In other words, $l(r_i w) < l(w)$. Since $s_i * A \subseteq \{\alpha_i\} \cup r_i(A)$, we obtain $\Phi_w \subseteq \{\alpha_i\} \cup r_i(A)$. Put $w' = b^{-1}(L(s_i g(A)))$. By Lemma 4.3, $\Phi_{w'}$ is the maximal element of $\mathcal{D}$ contained in $\{\alpha_i\} \cup r_i(A)$ and, by Lemma 4.4, so is Φ_w. Therefore, by Lemma 2.2(vi), $w = w'$, proving $g(s_i * A) = b(w) = L(s_i g(A))$. ∎

For $x \in \Omega$, write

$$C_x = \bigcup_{A \in \mathcal{C}, g(A) = x} U_A. \tag{19}$$

PROPOSITION 4.6: *The subsets C_x ($x \in \Omega$) satisfy the following three properties for each $x, y \in \Omega$:*

(i) *$C_x \neq \emptyset$.*
(ii) *$C_x \cap C_y = \emptyset$ if $x \neq y$.*
(iii) *$xC_y \subset C_{L(xy)}$.*

Proof: (i) Clearly, $\emptyset \neq U_{\Phi_{b^{-1}(x)}} \subseteq C_x$, so C_x is nonempty.

(ii) This follows immediately from the definition of C_x.

(iii) Given $x, y \in \Omega$, let $A \in \mathcal{C}$ be such that $y = g(A)$. Then, by respectively the definition of $*$, the definition of C_x, and Proposition 4.5(vi),

$$xU_A \subseteq U_{x*A} \subseteq C_{g(x*A)} = C_{L(xy)},$$

whence $xC_y \subseteq C_{L(xy)}$. ∎

In fact, (iii) also holds for each $x \in B^+$, as follows from the following argument based on induction with respect to $l(x)$. If $l(x) > 1$ then there exist $i \in \{1, \ldots, n\}$ and $u \in B^+$ such that $x = s_i u$ and $l(x) = 1 + l(u)$. Then, by the induction hypothesis, (iii) of the proposition, and Proposition 2.1,

$$xC_y = s_i u C_y \subseteq s_i C_{L(uy)} \subseteq C_{L(s_i L(uy))} = C_{L(s_i uy)} = C_{L(xy)}.$$

PROPOSITION 4.7: *Let B^+ act on a set U in such a way that each element acts injectively. Suppose we are given subsets C_x of U for $x \in \Omega$ satisfying properties* (i), (ii), *and* (iii) *of Proposition 4.6. Then the action of B^+ on U is faithful.*

Proof: (This is the proof appearing in [6].) Suppose that the elements x and y of B^+ act identically on U. If $l(x) + l(y) = 0$, then x and y are both the identity and there is nothing to prove. Suppose, therefore, that $l(x) + l(y) > 0$. Pick $u \in C_1$. Then $xu \in xC_1 \cap yC_1 \subseteq C_{L(x)} \cap C_{L(y)}$, which implies by Proposition 4.6 that $z = L(x) = L(y)$ for some nontrivial $z \in \Omega$. This means that there are x', y' in B^+ such that $x = zx'$ and $y = zy'$. But then, as z acts injectively, x' and y' act identically on U, whereas $l(x') + l(y') = l(x) + l(y) - 2l(z)$, so we can finish by induction on $l(x) + l(y)$. ∎

Proofs of Theorems 1.1 and 1.2: Propositions 4.6 and 4.7 with U as in (18) and C_x as in (19), together with Corollary 3.7, give a proof of Theorem 1.2. As for Theorem 1.1, suppose that M is of finite type. If M is the disjoint union of diagrams M' and M'', then B is the direct product of the Artin groups B', B'' corresponding to M', M'', respectively, and so the direct sum of faithful linear representations of B' and B'' would be a faithful linear representation for B. Hence, a proof of Theorem 1.1 in the case where M is finite and irreducible suffices.

By [4], every Artin group B of finite type M such that M has a multiple bond occurs as a subgroup of an Artin group of finite type without multiple bonds. Therefore, a proof of Theorem 1.1 for finite irreducible types without multiple bonds, that is, for types A, D, E, suffices, and this is dealt with by Theorem 1.2. This ends the proof of the theorems in Section 1.

5. Epilog

As stated before, the Artin groups whose types are spherical irreducible Coxeter matrices with multiple bonds occur as subgroups of Artin groups of finite types without multiple bonds. They occur as fixed subgroups of an automorphism group H of B permuting the vertices of M. The natural generators of this subgroup are the elements $\prod_{k \in E} s_k$ of B for E running over the H orbits on the vertex set of M. It is obvious that these subgroups satisfy the Artin group relations, but it is harder to establish that every relation they satisfy is a consequence of these. It may be of interest to know whether the latter can also be proved by applying Krammer's methods to the representation of the H fixed subgroup of B on the centralizer in V of H with respect to a suitable action of H on V.

We are also able to recover Theorem 6.1 from [6]. For this we need the **Charney length function** l_Ω on B. It assigns to $x \in B$ the smallest natural number k such that there are elements $x_1, x_2, \ldots, x_k$ in $\Omega \cup \Omega^{-1}$ for which $x = x_1 x_2 \cdots x_k$.

THEOREM 5.1: *Let B, R, V be as in Theorem 1.2 and write ρ for the linear representation $B \to \mathrm{GL}(V)$. For $x \in B$, consider the Laurent expansion of $\rho(x)$ with respect to t:*

$$\rho(x) = \sum_{i=k}^{h} A_i t^i, \quad A_k \neq 0, \quad A_h \neq 0,$$

where A_i is a matrix whose entries are in $\mathbb{Z}[r^{\pm 1}]$.

(i) *Then $l_\Omega(x) = \max(h-k, h, -k)$.*

(ii) *If in addition $x \in B^+ \setminus b(w_0)B^+$, then $k = 0$ and $h = l_\Omega(x)$. Here w_0 is the longest word in the Coxeter group W corresponding to B.*

Proof: The proof is as in [6] and so we do not include it. The use of Lemmas 3.1 and 3.2 in [6] is replaced by the following corresponding results for ρ.

There is a linear transformation $U \in \mathrm{GL}(V)$ whose matrix with respect to $\{x_\beta\}_\beta$ has entries in $\mathbb{Z}[r^{\pm 1}]$ such that $\sigma_k U \widehat{\sigma}_k = U$ for each $k \in \{1, \ldots, n\}$, where $\widehat{\sigma}_k$ is the matrix σ_k with t and r replaced by t^{-1} and r^{-1}, respectively. The matrix U is determined by the following rules involving an index $k \in \{1, \ldots, n\}$ such that $(\alpha_k, \beta) = 1$:

$$U_{\gamma,\beta} = \begin{cases} 0 & \text{if } \gamma \not\leq \beta, \\ 1 & \text{if } \gamma = \beta, \\ \widehat{T}_{k,\beta} r^4 & \text{if } \gamma = \alpha_k \leq \beta, \\ U_{\gamma-\alpha_k, \beta-\alpha_k} & \text{if } \gamma \leq \beta \text{ and } (\alpha_k, \gamma) = 1, \\ r^{-1} U_{\gamma, \beta-\alpha_k} & \text{if } \gamma \leq \beta \text{ and } (\alpha_k, \gamma) = 0, \\ U_{\gamma+\alpha_k, \beta-\alpha_k} + (r^{-1} - r) U_{\gamma, \beta-\alpha_k} & \text{if } \gamma \leq \beta \text{ and } (\alpha_k, \gamma) = -1. \end{cases}$$

This matrix replaces the matrix $T(q)$ in Lemma 3.1 in [6].

In the representation of Theorem 1.2, $\rho(b(w_0))$ is the multiple of the permutation matrix π by the scalar tr^{e+3}. Here π permutes $\{x_\beta\}_\beta$ according to the action of $-w_0$ on Φ^+ and e is the number of positive roots that are not orthogonal to a given root. In particular $e + 3 = 2(n+1)$ for A_n, $4(n-1)$ for D_n, 24 for E_6, 36 for E_7, and 60 for E_8. Note that this is in accordance with the theorem we are proving as $k = h = 1$. The matrix $\rho(b(w_0))$ replaces the matrix of Lemma 3.2 in [6]. ∎

Just as in [6], this leads to a different proof that ρ is faithful. Indeed, if x is in the kernel, we see $h = k = 0$ and so $l_\Omega(x) = 0$, establishing that x is the unit element of B.

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